

# Differential equations for definition and evaluation of Feynman integrals

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## Abstract

It is shown that every Feynman integral can be interpreted as Green function of some linear differential operator with constant coefficients. This definition is equivalent to usual one but needs no regularization and application of  $R$ -operation. It is argued that presented formalism is convenient for practical calculations of Feynman integrals.

Though fundamental results in renormalization theory were obtained many years ago in classical works of Feynman, Tomonaga, Schwinger, Dyson, Salam, Bogolubov, Parasiuk, Hepp, and Zimmermann<sup>1</sup>, renormalization problems continue to attract the attention of theorists. In particular, during last twenty years very many papers were devoted to investigations of various regularization schemes.

Of course, all known regularization schemes are equivalent, in principal, at perturbative level. However, their practical value is different. For instance, only the discovery of dimensional regularization [4] gave possibility to carry out systematical calculations in

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<sup>1</sup>Beautiful account of foundations and modern achievements of renormalization theory can be found, for instance, in monographs [1, 2, 3]

gauge theories. Moreover, different regularization schemes, that are equivalent on perturbative level, can be nonequivalent beyond perturbation theory. For instance, partial summing of perturbation series by means of renormalization group methods can give scheme dependent results (see, for instance, [6] .) This fact stimulates further search of "the most natural" and convenient regularization scheme.

In this paper we will show that Feynman integrals can be defined and evaluated without any regularization at all. Of course, in itself it is not a surprise. In particular, recently proposed differential regularization [5] also needs no regularization in usual sense. But simplicity of our results is the real surprise. *We will show that any Feynman diagram without internal vertexes can be treated as Green function of some linear differential operator with constant coefficients.* This result allows also to define and evaluate Feynman diagram with internal vertexes, because such diagrams can be considered as certain diagrams without internal vertexes at zero value of some external momenta. For instance, the value of diagram with internal vertexes on Fig.1 coincides with the value at  $k = q = 0$  of diagram on Fig.2 .

Renormalization scheme, given in this paper, is equivalent to usual R-operation scheme. But "equivalent" doesn't mean "the same". Indeed, in standard R-operation renormalization scheme one must, first, regularize initial divergent (in general) Feynman integral. Then it is necessary to use rather complicated subtraction prescription (forest formula) to obtain finite result. Nothing similar is needed in my renormalization scheme. To obtain finite expression for given Feynman integral, one must only solve some well defined differential equations. Neither any regularization, nor any manipulations with counterterm diagrams are needed to obtain finite result.

For simplicity, in this paper we will consider only scalar Feynman integrals. General case will be investigated in more long forthcoming paper.

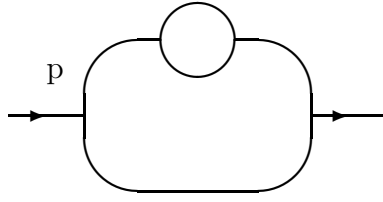


Figure 1: Example of a diagram with internal vertexes

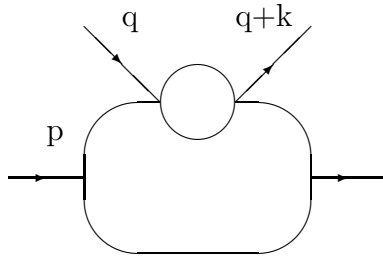


Figure 2: The diagram without internal vertexes that corresponds to one on Fig.1

Let us consider arbitrary (Euclidean) Feynman diagram without internal vertexes in coordinate space. This is well defined function

$$\tilde{\Gamma}_n(x_1, \dots, x_n; \{m_{ij}^2\}) = \prod_{\text{all lines of } \Gamma} D(x_i - x_j; m_{ij}^2) \quad (1)$$

where

$$D(x, m^2) = \int d^4p \frac{e^{ipx}}{p^2 + m^2}$$

But their Fourier image

$$\Gamma_n(p_1, \dots, p_{n-1}; \{m_{ij}^2\}) = \frac{1}{(2\pi^4)^{n-1}} \int d^4x_1 \dots d^4x_{n-1} \exp(ip_1x_1 + \dots + ip_{n-1}x_{n-1}) \tilde{\Gamma}(x_1, \dots, x_{n-1}, 0; \{m_{ij}^2\}) \quad (2)$$

is not, in general, well defined. The problem of renormalization theory is to define the function  $\Gamma(p_1, \dots, p_{n-1}; \{m_{ij}^2\})$ .

Below we will interpret  $m_{ij}^2$  as the square of some Euclidean two dimensional vector. Then we can write

$$m_{ij}^2 = (m_{ij,1})^2 + (m_{ij,2})^2$$

Further, let us define the Fourier image of  $D(x, m^2)$  with respect to variables  $m_1$  and  $m_2$  (here  $m_1^2 + m_2^2 = m^2$ ):

$$\hat{D}(x, u) = \int d^2m e^{i\vec{u}\vec{m}} D(x, m) = \int d^4p d^2m \frac{e^{ipx + i\vec{u}\vec{m}}}{p^2 + m^2}$$

where  $\vec{u} = (u_1, u_2)$ ,  $u^2 = u_1^2 + u_2^2$ . It follows from definition that  $\hat{D}(x, u)$  is the Green function of six dimensional Laplace operator

$$\Delta_{xu} = \sum_{i=1}^4 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^2 \frac{\partial^2}{\partial u_i^2}$$

So

$$\hat{D}(x, u) = 16\pi^3 \frac{1}{(x^2 + u^2)^2}$$

and

$$\begin{aligned} \hat{\Gamma}_n(x_1, \dots, x_n; \{u_{ij}^2\}) &\equiv \int \prod_{\text{all lines of } \Gamma} d^2 m_{ij} \exp \left( i \sum_{\text{all lines of } \Gamma} \vec{m}_{ij} \vec{u}_{ij} \right) \tilde{\Gamma}_n(x_1, \dots, x_n; \{m_{ij}^2\}) \\ &= (16\pi^3)^N \frac{1}{P(x_1, \dots, x_n; \{u_{ij}^2\})} \end{aligned}$$

where  $N$  is total number of lines in diagram  $\Gamma$  and  $P$  is the polynomial:

$$P = \prod_{\text{all lines of } \Gamma} [(x_i - x_j)^2 + u_{ij}^2]^2 \quad (3)$$

We see that  $\hat{\Gamma}$  satisfies simple algebraic equation

$$P(x_1, \dots, x_n; \{u_{ij}^2\}) \hat{\Gamma}(x_1, \dots, x_n; \{u_{ij}\}) = (16\pi^3)^N \quad (4)$$

Comparing (2), (3) and (4), we see that it is very naturally *to define*  $\Gamma(p_1, \dots, p_{n-1})$  as a solution of differential equation

$$\begin{aligned} P \left( i \frac{\partial}{\partial p_1}, \dots, i \frac{\partial}{\partial p_{n-1}}, 0; \{\Delta_{m_{ij}}\} \right) \Gamma_n(p_1, \dots, p_{n-1}; \{m_{ij}^2\}) \\ = (16\pi^3)^N \delta(p_1) \dots \delta(p_{n-1}) \prod_{\text{all lines of } \Gamma} \delta(\vec{m}_{ij}) \end{aligned} \quad (5)$$

This means that  $\Gamma$  is Green function of linear differential operator  $P(i\partial/\partial p_1, \dots)$ . For instance, the diagram on Fig.2 is defined by equation

$$(\Delta_{pm_1})^2(\Delta_{(p-q)m_2})^2(\Delta_{km_3})^2(\Delta_{qm_4})^2(\Delta_{qm_5})^2\Gamma = (16\pi^3)^5\delta(p)\delta(q)\delta(k)\prod_{i=1}^5\delta(\vec{m}_i)$$

where

$$\Delta_{(p-q)m_2} = \sum_{i=1}^4 \left( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial q_i} \right)^2 + \sum_{i=1}^2 \frac{\partial^2}{\partial m_{2,i}^2}$$

Eq.(5) defines  $\Gamma$  up to solution of homogeneous equation

$$P \left( i \frac{\partial}{\partial p_1}, \dots, i \frac{\partial}{\partial p_{n-1}}, 0; \{\Delta_{m_{ij}}\} \right) \Gamma = 0$$

This arbitrariness can be fixed inductively in the following way.

Let all diagrams with  $(L-1)$  loops are already defined. For given  $L$ -loop diagram with divergent index  $\omega(\Gamma)$  one defines  $(L-1)$ -loop diagram  $\Gamma_{ij}$  as diagram  $\Gamma$  without the line  $(ij)$  with propagator  $((p-k)^2 + m_{ij}^2)^{-1}$  where  $p$  is external momentum. We can always define external momenta in such way that  $\Gamma_{ij}$  doesn't depend on  $p$ . (See Fig.3, where  $\Gamma_{ij}$  is represented as shaded block.)

It is easy to see that if  $\Gamma$  satisfies the equations

$$(\Delta_{pm_{ij}})^2\Gamma = (16\pi^3)\delta(\vec{m}_{ij})\Gamma_{ij} \quad (6)$$

then  $\Gamma$  also satisfies the equation (5). Finally, we impose asymptotic conditions

$$\lim_{|p| \rightarrow \infty} \frac{1}{p^{\omega(\Gamma)+\epsilon}} \Gamma = 0; \quad \lim_{m_{ij} \rightarrow \infty} \frac{1}{m_{ij}^{\omega(\Gamma)+\epsilon}} \Gamma = 0; \quad (7)$$

for any  $\epsilon > 0$ .

It is easy to prove that equations (6) together with analogous equations for other lines with asymptotic conditions (7) define  $\Gamma$  up to polynomial of degree  $\omega(\Gamma)$  with respect

to external momenta and masses. (For  $\omega(\Gamma) < 0$  the diagram  $\Gamma$  is defined unequivocally if  $(L - 1)$ -loop diagrams are already defined.) So our definition reproduces usual renormalization arbitrariness in the definition of Feynman integrals.

Now let us prove that for renormalizable theories with divergent index less or equal two our definition is equivalent to usual one. Consider again the diagram on Fig.3 regularized by means of the cut off at large momentum  $\Lambda$ . We will denote this diagram as  $\Gamma_\Lambda$ . The renormalized diagram  $\Gamma^{ren}(\Lambda)$  is the sum of  $\Gamma_\Lambda$  and counterterm diagrams. The later ones can be divided in two sets. First set of counterterm diagrams contains the line  $(ij)$ . The sum of  $\Gamma_\Lambda$  and these diagrams can be written as

$$\int_{|k|<\Lambda} d^4k \frac{1}{(p-k)^2 + m_{ij}^2} \Gamma_{ij}^{ren}(\Lambda) \quad (8)$$

where  $\Gamma_{ij}^{ren}(\Lambda)$  is renormalized diagram  $\Gamma_{ij}$ . This diagram doesn't depend on  $p$ .

The second set of counterterm diagrams doesn't contain the line  $(ij)$ . They are produced by change of some divergent subdiagrams of  $\Gamma$ , that contain the line  $(ij)$ , on polynomials not more then the second degree with respect to external momenta and masses (See, for instance [1, 2, 3]). In particular, these polynomials are not more then the second degree with respect to  $p$  and  $m_{ij}$ . The diagram  $\Gamma^{ren}(\Lambda)$  is the sum of (8) and these polynomials.

Using the formulae

$$(\Delta_{pm_{ij}})^2 \frac{1}{(p-k)^2 + m_{ij}^2} = 16\pi^3 \delta(p-k) \delta(\vec{m}_{ij}) \quad (9)$$

and (8), one can prove that

$$(\Delta_{pm_{ij}})^2 \Gamma^{ren}(\Lambda) = 16\pi^3 \delta(\vec{m}_{ij}) \theta(\Lambda - |p_i|) \Gamma_{ij}^{ren}(\Lambda) \quad (10)$$

(because above mentioned polynomials are annihilated by  $(\Delta_{pm_{ij}})^2$ .) In the limit  $\Lambda \rightarrow \infty$  one obtains the equation (6). Asymptotic conditions (6) are satisfied due to Weinberg's theorem [8] . This finishes the prove.

Now let us consider an illustrative example that show how our definition works in practical calculations.

The diagram  $\Gamma^{(1)}$  on Fig.4 is the simplest divergent Feynman one. In according to our general theory it is defined by equations

$$(\Delta_{pm_1})^2 \Gamma^{(1)} = \frac{16\pi^3 \delta(\vec{m}_1)}{p^2 + m_2^2} \quad (11)$$

$$(\Delta_{pm_2})^2 \Gamma^{(1)} = \frac{16\pi^3 \delta(\vec{m}_2)}{p^2 + m_1^2} \quad (12)$$

Using the formula <sup>2</sup>

$$\frac{1}{p^2 + m_2^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s) \Gamma(1-s) \frac{(m_2^2)^{s-1}}{(p^2)^s} \quad (13)$$

$0 < \text{Re } c < 1$

one can represent the solution of (11) in the form

$$\Gamma^{(1)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s) \Gamma(1-s) (m_2^2)^{s-1} \Gamma_s + f_1(m_2^2) \quad (14)$$

$0 < \text{Re } c < 1$

where  $\Gamma_s$  satisfies the equation

$$(\Delta_{pm_1})^2 \Gamma_s = \frac{16\pi^3 \delta(\vec{m}_1)}{(p^2)^s} \quad (15)$$

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<sup>2</sup>Representation (13) was used first for evaluation of Feynman integrals in [9]



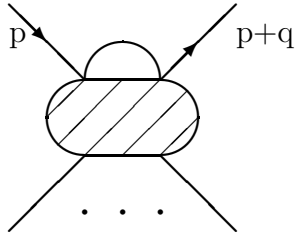


Figure 3: Illustration to the proof of equivalence of proposed renormalization scheme and usual one

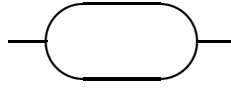


Figure 4: The simplest divergent Feynman diagram



Figure 5: "Raising sun" diagram

and  $f_1(m_2^2)$  is arbitrary function.

If in (15)  $1 < \text{Re } s < 2$ , then the solution of (15) is convergent Feynman integral

$$\Gamma_s = \int d^4k \frac{1}{[(p-k)^2 + m_1^2](k^2)^s} \quad (16)$$

(This can be proved by using of formula (9).)

Introducing Feynman parameters, one can write  $\Gamma_s$  in the form

$$\Gamma_s = \frac{\pi^2}{s-1} \int_0^1 dx \left[ \frac{x}{(1-x)(xp^2 + m_1^2)} \right]^{s-1} \quad (17)$$

We see that  $\Gamma_s$  can be analytically continued in the strip  $0 < \text{Re } s < 1$  before the integration with respect to  $x$ . Substituting (17) in (14) and integrating with respect to  $s$ , one obtains:

$$\Gamma^{(1)} = -\pi^2 \int_0^1 dx \ln \left( 1 + \frac{(1-x)(xp^2 + m_1^2)}{xm_2^2} \right) + f_1(m_2^2) \quad (18)$$

Let  $\mu^2$  is arbitrary constant with dimension of  $[\text{mass}]^2$ . Then (18) can be rewritten in the form

$$\Gamma^{(1)} = \left\{ -\pi^2 \int_0^1 dx \ln \left( \frac{x(1-x)p^2 + xm_1^2 + (1-x)m_2^2}{\mu^2} \right) \right\} + \pi^2 \left( 1 + \ln \frac{\mu^2}{m_2^2} \right) + f_1(m_2^2) \quad (19)$$

Equation (12) can be solved in the analogous way. Comparing results, one can prove that the sum of terms out of braces in (19) is constant. It can be included in the definition of  $\mu^2$ . So, finally, we obtain familiar result:

$$\Gamma^{(1)}(p^2, m_1^2, m_2^2) = -\pi^2 \int_0^1 dx \ln \left( \frac{x(1-x)p^2 + xm_1^2 + (1-x)m_2^2}{\mu^2} \right)$$

$$= \pi^2 \left\{ \ln \frac{\mu^2}{m_1 m_2} - \frac{(m_1^2 - m_2^2)}{2p^2} \ln \frac{m_2^2}{m_1^2} + f \ln \frac{p^2 + m_1^2 + m_2^2 - 2p^2 f}{p^2 + m_1^2 + m_2^2 + 2p^2 f} \right\} \quad (20)$$

where

$$f = \sqrt{\frac{(m_1^2 - m_2^2)^2}{4p^4} + \frac{m_1^2 + m_2^2}{2p^2} + \frac{1}{4}}$$

We see that finite result for divergent diagram on Fig.4 can be obtained without any regularization and application of  $R$ -operation.

Now let us consider less trivial application of our theory. We will calculate two-loop diagram on Fig.5. In general, this diagram depends on three different masses  $m_1, m_2, m_3$ . For simplicity, we will consider only the most important case  $m_1 = m_2 \equiv m$ ,  $m_3 \equiv M$ . General case can be treated analogously.

Power expansions for this diagram were investigated for equal mass case in [10] and for general case in recent work [11]. See also [12] for corresponding numerical results.

Up to a polynomial of the first degree with respect to  $p^2$ , the corresponding Feynman integral can be defined by equation:

$$(\Delta_{pM})^2 \Gamma^{(2)}(p^2, M^2, m^2) = 16\pi^3 \delta(\vec{M}) \Gamma^{(1)}(p^2, m^2, m^2) \quad (21)$$

Up to unessential constant,  $\Gamma^{(1)}$  can be represented in the following form:

$$\Gamma^{(1)}(p^2, m^2, m^2) = \pi^2 \int_{4m^2}^{\infty} d\sigma^2 \sqrt{1 - \frac{4m^2}{\sigma^2}} \left( \frac{1}{p^2 + \sigma^2} - \frac{1}{\sigma^2} \right) \quad (22)$$

Comparing (11), (21) and (22), we see that  $\Gamma^{(2)}$  can be represented as following:

$$\Gamma^{(2)}(p^2, M^2, m^2) = \pi^2 \int_{4m^2}^{\infty} d\sigma^2 \sqrt{1 - \frac{4m^2}{\sigma^2}} \left( \Gamma^{(1)}(p^2, M^2, \sigma^2) - \frac{1}{\sigma^2} \varphi(p^2, M^2, \sigma^2) \right) \quad (23)$$

where  $\Gamma^{(1)}(p^2, M^2, \sigma^2)$  is defined by (20) and  $\varphi(p^2, M^2, \sigma^2)$  satisfies the equation

$$(\Delta_{pm})^2 \varphi = 16\pi^3 \delta(\vec{M}) \quad (24)$$

Using the formula

$$\Delta_M \ln M^2 = 2\pi \delta(\vec{M})$$

one can represent the solution of (24) in the following form:

$$\varphi = \pi^2 \left\{ M^2 \ln \frac{M^2}{\sigma^2} + M^2 f_1(\sigma^2) + p^2 f_2(\sigma^2) + f_3(\sigma^2) \right\} \quad (25)$$

where  $f_1, f_2, f_3$  are arbitrary functions of  $\sigma^2$ . These functions must be defined thus that integral (23) converges. Using explicit formula (20) for  $\Gamma^{(1)}$ , it is easy to prove that one of the possible choices is

$$f_1 = 0, \quad f_2 = -\frac{1}{2}, \quad f_3 = \sigma^2 \left( \ln \frac{\mu^2}{\sigma^2} - 1 \right) \quad (26)$$

The replacement of functions  $f_1, f_2, f_3$  by any other ones, for which integral (22) converges, leads to unessential change of  $\Gamma^{(2)}$  on the polynomial of the first degree with respect to  $p^2$ .

Substituting of (25) and (26) in (23), we obtain our final result:

$$\Gamma^{(2)} = \pi^2 \int_{4m^2}^{\infty} d\sigma^2 \sqrt{1 - \frac{4m^2}{\sigma^2}} \left( \Gamma^{(1)}(p^2, M^2, \sigma^2) - \pi^2 \frac{M^2}{\sigma^2} \ln \frac{M^2}{\sigma^2} - \pi^2 \frac{p^2}{2\sigma^2} - \pi^2 \ln \frac{\mu^2}{\sigma^2} + \pi^2 \right) \quad (27)$$

The integrand in (27) has the order  $O(\sigma^{-4} \ln \sigma^2)$  at  $\sigma^2 \rightarrow \infty$ . So integral (27) converges.

To author's knowledge , integral (27) can't be expressed through standard special functions. But integrand in (23) is rather simple elementary function and so this formula makes possible to investigate  $\Gamma^{(2)}$  in details. This will be done in special paper.

One-fold integral representation, that is very similar to (27), was obtained independently in works [11] and [12] by dispersive methods. Analogous representation for five propogators selfenergy diagram can be found in [13, 14].

Now it is unclear, whether our approach to renormalization theory has principal advantages in comparison with standard formulation. But, at least, calculations, represented above, show that our approach gives new effective methods of evaluation of Feynman integrals. So author believes that proposed formalism will be useful in various investigations in quantum field theory.

Author is indebted to D.J.Broadhurst, A.I.Davydychev and V.A.Smirnov for valuable discussions and comments and also F.A.Berends, M.Buza, and J.B.Tausk for sending preprints of their recent works.

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